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Multidimensional Strong Large Deviation Theorems

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September 1992

Abstract

We obtain a strong large deviation result for an arbitrary sequence of random vectors under simple and verifiable conditions on the moment generating functions. The key to this result is a local limit theorem for arbitrary sequences of random vectors which is also proved in this paper. The local limit theorem gives conditions on the characteristic functions of random vectors for their pseudo-density function to converge uniformly on bounded sets. We apply these results to the multivariate F-distribution.

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Multidimensional Strong Large Deviation Theorems

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1 Introduction.

Let $\{T_n, n \geq 1\}$ be a sequence of random vectors taking values in R^k, such that $T_n/n \to c$ in probability as $n \to \infty$. When the closure of the set S does not contain c, the probability of the event $\{T_n/n \in S\}$ tends to 0 and oftentimes at an exponential rate. Under these circumstances the event $\{T_n/n \in S\}$ is called a large deviation event. Large deviation theorems provide asymptotic expressions for the logarithm of probability of the large deviation events while strong large deviation theorems provide asymptotic expressions for the probability of the large deviation events. In this paper we will be mainly concerned with strong large deviation results.

In the univariate case, when T_n is the sum of n independent random variables with a common nonlattice distribution function (d.f.) F, Cramér (1938) established one of the earliest strong large deviation theorems for $P(T_n/n > m)$. The lattice case was treated by Blackwell and Hodges (1959). Bahadur and Ranga Rao (1960) generalized these results by improving the remainder terms, both for nonlattice and lattice d.f. F under the assumption of finiteness of the moment generating function (m.g.f.) of X_1 in an interval around the origin. When the m.g.f. of X_1 is finite in some nondegenerate interval, Petrov (1965) obtained analogous results which hold uniformly for m in a compact interval with zero as one of its end points. Höglund (1979) presented a unified formulation of the results of Bahadur and Ranga Rao (1960) and Petrov (1965), both for small and large deviations. He gave conditions under which the approximation holds not only when m belongs to compact sets but also when m is close to ∞.

A k-dimensional result which is related to the continuous case of Höglund (1979) was given by Borovkov and Rogozin (1965). These authors obtained strong large deviation theorems for $P(T_n/n \in S_n)$ under minimum conditions on the measurable sets

$\{S_n,\, n \geq 1\}$. Von Bahr (1967) obtained strong large deviation theorems for measurable sets $\{S_n,\, n \geq 1\}$ which can be written as the difference of two convex sets considering two cases. In the first case S_n's were assumed to be a subset of a sphere with its center at the origin and in the other case S_n's were assumed to be contained in the complement of such a sphere. In a recent paper Robinson, et. al. (1990) extended the results of Höglund (1979). They considered the case where T_n is the sum of n independent random vectors in R^k with the first $k_0 \leq k$ dimensions being lattice with span 1.

In this paper we obtain strong large deviation theorems for arbitrary sequences of random vectors, under simple and easily verifiable conditions on the moment generating functions. These results for arbitrary random vectors, generalize the univariate results of Chaganty and Sethuraman (1992) and those of Petrov (1965). The key to the strong large deviation result is a local limit theorem for arbitrary sequences of random vectors, that is, a theorem on the convergence of pseudo-densities. Before obtaining the strong large deviation result, we will prove such a local limit theorem under mild conditions on the characteristic functions (c.f.'s) of the random vectors.

Let $\{Y_n,\, n \geq 1\}$ be a sequence of random vectors which converges in distribution to Y. For any set S, let S^0 be the interior of S and $\overline{S}$ be the closure of S. Let S be a measurable subset of R^k such that $0 < \mu(S^0) = \mu(\bar{S}) < \infty$, where μ is the Lebesgue measure on R^k and $b_n \to \infty$ be a sequence of real numbers. Since Y_n may not possess a p.d.f., we will use $q_n(y;\, b_n, S) = b_n^k\; P(b_n(Y_n - y) \in S)/\mu(S)$ and call it the pseudo-density of Y_n. The convergence of $q_n(y;\, b_n, S)$ to the probability density function of Y will be referred to as a local limit theorem. In this paper we also obtain two local limit theorems (Theorems 2.1 and 2.2) for random vectors Y_n under different conditions on the c.f. of Y_n. A key result concerning asymptotic expressions for the Laplace transform on the set $Y_n \geq 0$ is proved in Theorem 2.4. This result is used later in Section 3.

As mentioned earlier, these local limit theorems are used to obtain a strong deviation result (Theorem 3.4) for arbitrary random vectors. As an application of this large deviation result, we obtain expressions for the tail probabilities of the k dimensional multivariate F-distribution in Section 4.

2 Multidimensional Local Limit Theorems.

Let $\{Y_n, n \geq 1\}$ be a sequence of random vectors taking values in R^k, which converges to a random vector Y in distribution. Let S be a measurable subset of R^k satisfying $0 < \mu(S^0) = \mu(\overline{S}) < \infty$, where μ is the Lebesgue measure on R^k. Let $b_n \to \infty$ be a sequence of real numbers. Define

$$q_n(y;\, b_n,\, S) \;=\; \frac{b_n^k}{\mu(S)} P(b_n(Y_n - y) \in S) \tag{2-1}$$

for y in R^k. Notice that $q_n(y;\, b_n,\, S)$ is the p.d.f. of $Y_n + Z_n$ where Z_n is distributed uniformly on $-S/b_n$. We will refer to it as the pseudo-density of Y_n and study its limiting properties, since Y_n itself may not possess a p.d.f. Let $\{y_n, n \geq 1\}$ be a sequence of random vectors in R^k such that $y_n \to y^*$. We will refer the convergence of $q_n(y_n;\, b_n,\, S)$ to the p.d.f. of Y at y^* as a local limit theorem. We will use the following notations to denote norms, vector products, etc. Let $t = (t_1, \ldots, t_k)$, $s = (s_1, \ldots, s_k)$ be vectors in R^k. Define

$$\begin{aligned}
< t, s > &= t_1 s_1 + \cdots + t_k s_k \\
\|t\| &= \max_{1 \leq j \leq k} |t_j| \\
|t| &= |t_1| + \cdots + |t_k| \\
t^s &= t_1^{s_1}\, t_2^{s_2} \ldots t_k^{s_k} \\
t &\geq s \quad \text{if } t_i \geq s_i \quad \forall \quad i.
\end{aligned} \tag{2-2}$$

We first obtain a local limit theorem under a strong condition. This is given in Theorem 2.1 below. This strong condition is relaxed and a more useful result is given in Theorem 2.2.

THEOREM **2.1** *Let $\{Y_n, n \geq 1\}$ be a sequence of random vectors which converges weakly to a random vector Y. Let $\hat{f}_n(t)$ and $\hat{f}(t)$ be the c.f.'s of Y_n and Y respectively. Suppose that there exists an integrable function $f^*(t)$ such that*

$$\sup_n |\hat{f}_n(t)| \leq f^*(t) \tag{2-3}$$

for all t. Then Y_n possesses a bounded and continuous p.d.f. $f_n(y)$, Y also possesses a bounded and continuous p.d.f. $f(y)$, and $f_n(y_n)$ converges $f(y^*)$ if $y_n \to y^*$.

Proof. Condition (2-3) implies that the c.f.'s $\hat{f}_n(t)$ and $\hat{f}(t)$ are integrable. Hence both Y_n and Y possess a bounded and continuous p.d.f.'s. The inversion formula and the dominated convergence theorem show that $f_n(y_n)$ converges to $f(y^*)$ if $y_n \to y^*$ as $n \to \infty$. ◇

Condition (2-3) is too strong to be useful in most situations. We show in Theorem 2.2 below that appropriate bounds on the c.f. $\hat{f}_n(t)$ on increasing sequences of bounded intervals are sufficient to obtain results similar to Theorem 2.1.

THEOREM **2.2** *Let $\{Y_n, n \geq 1\}$ be a sequence of random vectors which converges to a random vector Y in distribution. Let $\hat{f}_n(t)$ be the c.f. of Y_n for $n \geq 1$ and let $\hat{f}(t)$ be the c.f. of Y. Let $\{\beta_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\beta_n \to \infty$ and $b_n \to \infty$. Suppose that there exists an integrable function $f^*(t)$ such that for $t \in R^k$,*

$$\sup_n |\hat{f}_n(t)|\, I(\|t\| \leq \beta_n) \leq f^*(t) \tag{2-4}$$

and

$$\theta_n(\lambda) \stackrel{\text{def}}{=} \sup_{\beta_n < \|t\| \le \lambda b_n} |\hat{f}_n(t)| \;=\; o\left(\frac{1}{b_n^k}\right) \tag{2-5}$$

for each $\lambda > 0$, where the above supremum is defined to be zero if $\{t : \beta_n < \|t\| \le \lambda b_n\}$ is empty. Then the random vector Y possesses a bounded and continuous p.d.f. $f(y)$. Let $q_n(y;\, b_n,\, S)$ be the pseudo-density function of Y_n as defined in (2-1). Then there exists a finite constant M and an integer n_s depending on S, such that

$$\sup_y \left[q_n(y;\, b_n,\, S)\right] \;\le\; M \tag{2-6}$$

for $n \ge n_s$. Furthermore, if $y_n \to y^*$ then

$$q_n(y_n;\, b_n,\, S) \;\to\; f(y^*) \tag{2-7}$$

as $n \to \infty$.

Proof: Since $\hat{f}_n(t) \to \hat{f}(t)$ pointwise and $\beta_n \to \infty$, condition (2-4) implies that $\hat{f}(t)$ is bounded by $f^*(t)$. Hence Y possesses a bounded and continuous p.d.f. $f(y)$. Suppose β_n/b_n is bounded. Since $b_n^k\theta_n(\lambda) \to 0$, for each $\lambda > 0$, we can find a sequence $\{\lambda_n\}$ satisfying

$$\lambda_n \to \infty \quad \text{and} \quad \lambda_n^k b_n^k \theta_n(\lambda_n) \to 0 \tag{2-8}$$

as $n \to \infty$. If $\beta_n/b_n \to \infty$, put $\lambda_n = \beta_n/b_n$ and in this case also (2-8) is satisfied because $\theta_n(\lambda_n) = 0$ for large n. For simplicity of notation set $\theta_n \stackrel{\text{def}}{=} \theta_n(\lambda_n)$. Let U_n be the uniform distribution on the set $-S/b_n$ and u_n, $\hat{u}_n$ be the p.d.f. and c.f. corresponding to U_n. We also introduce another distribution function (d.f.) V_n with p.d.f. v_n and c.f. $\hat{v}_n$, to obtain the important identity (2-11):

$$v_n(x) = \frac{\lambda_n^k b_n^k}{(2\pi)^k} \prod_{j=1}^{k} \left[\frac{\sin(\lambda_n b_n x_j/2)}{(\lambda_n b_n x_j/2)} \right]^2, -\infty < x_j < \infty,\, j = 1, \ldots, k \tag{2-9}$$

$$\hat{v}_n(t) = \begin{cases} \prod_{j=1}^{k} \left(1 - \frac{|t_j|}{\lambda_n b_n}\right) & \text{if } \|t\| \le \lambda_n b_n \\ 0 & \text{otherwise.} \end{cases} \tag{2-10}$$

Let F_n be the d.f. of Y_n, and let $Q_n = F_n * U_n$, $M_n = Q_n * V_n$ where $*$ denotes the convolution operation. Notice that $q_n(y;\, b_n,\, S)$ defined in (2-1) is the p.d.f. of Q_n. Let $m_n(y)$ be the p.d.f. of M_n. The c.f. $\hat{m}_n(t)$ of M_n, which is equal to $\hat{f}_n(t)\hat{u}_n(t)\hat{v}_n(t)$, vanishes outside the rectangle $\{t \in R^k : \|t\| \le \lambda_n b_n\}$. The inversion theorem yields the following identity:

$$\begin{aligned} m_n(y) &= \frac{1}{(2\pi)^k} \int \cdots \int_{\|t\| \le \lambda_n b_n} \exp(-i < t,\, y >)\, \hat{m}_n(t)\, dt_1 \ldots dt_k \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} q_n(y - x;\, b_n,\, S)\, v_n(x)\, dx_1 \ldots dx_k \\ &= \frac{b_n^k}{\mu(S)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(b_n(Y_n - y + x) \in S)\, v_n(x)\, dx_1 \ldots dx_k. \end{aligned} \tag{2-11}$$

Relation (2-11) is the starting point of the main part of this proof and it relates $q_n(y;\, b_n,\, S)$ to the integrable c.f. $\hat{m}_n(t)$. Let $y_n \to y^*$. We first show that $m_n(y_n)$ converges to $f(y^*)$ and then obtain lower and upper bounds for $m_n(y_n)$ which depend on $q_n(y_n;\, b_n,\, S)$. This will then establish (2-6) and (2-7). Using (2-8) and (2-11) we get

$$\begin{aligned} m_n(y) &= \frac{1}{(2\pi)^k} \int \cdots \int_{\|t\| \le \beta_n} \exp(-i < t,\, y >)\, \hat{m}_n(t)\, dt_1 \ldots dt_k \\ &\quad + \frac{1}{(2\pi)^k} \int \cdots \int_{\beta_n < \|t\| \le \lambda_n b_n} \exp(-i < t,\, y >)\, \hat{m}_n(t)\, dt_1 \ldots dt_k \\ &\le \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(t)\, dt_1 \ldots dt_k + \frac{\lambda_n^k b_n^k \theta_n}{(2\pi)^k} \end{aligned}$$

$$\leq \frac{2}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(t)\, dt_1 \ldots dt_k. \tag{2-12}$$

for $n \geq n_0$, where n_0 is independent of y. From condition (2-4), (2-8), (2-12) and the dominated convergence theorem and the inversion formula we get

$$m_n(y_n) \to \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-i < t, y^* >)\, \hat{f}(t)\, dt_1 \ldots dt_k = f(y^*). \tag{2-13}$$

Let $\eta > 0$. Let $S(x, \eta) = \{y : \|y - x\| \leq \eta\}$ be a ball of radius η (with norm $\|\cdot\|$) centered at x. Let $S_\eta = \{x : S(x, \eta) \subset S\}$ and $S^\eta = \{y : \|x - y\| \leq \eta$ for some $x \in S\}$. Since we have assumed that $\mu(S^0) = \mu(\overline{S})$ we can find $\eta(= \eta_s) > 0$ such that

$$\mu(S_\eta) > 0 \qquad \text{and} \qquad [\mu(S^\eta)/\mu(S)] \leq 2. \tag{2-14}$$

Note that $y \in S_\eta$ implies $y + x \in S(y, \eta) \subset S$ if $\|x\| < \eta$. From this, we get a lower bound for $m_n(y)$ as follows:

$$\begin{aligned} m_n(y) &\geq \frac{b_n^k}{\mu(S)} \int \cdots \int_{\|x\| < \eta/b_n} P(b_n(Y_n - y + x) \in S)\, v_n(x)\, dx_1 \ldots dx_k. \\ &\geq \frac{b_n^k}{\mu(S)} P(b_n(Y_n - y) \in S_\eta) \int \cdots \int_{\|x\| < \eta/b_n} v_n(x)\, dx_1 \ldots dx_k \\ &\geq \frac{b_n^k}{\mu(S)} P(b_n(Y_n - y_n) \in S_\eta) \left[1 - \frac{4}{\pi \lambda_n \eta}\right]^k. \end{aligned} \tag{2-15}$$

Using (2-12), (2-16) and (2-8) we get

$$\begin{aligned} &\frac{b_n^k}{\mu(S)} P(b_n(Y_n - y_n) \in S_\eta) \left[1 - \frac{4}{\pi \lambda_n \eta}\right]^k \\ \leq\ & m_n(y) \leq \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(t)\, dt_1 \ldots dt_k + \frac{\lambda_n^k b_n^k \theta_n}{(2\pi)^k} \\ \leq\ & \frac{2}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(t)\, dt_1 \ldots dt_k. \end{aligned} \tag{2-16}$$

for sufficiently large n. By replacing S by S^η and using (2-15) and the fact that $S \subset (S^\eta)_\eta$, we get

$$\begin{aligned}
&\frac{b_n^k}{\mu(S)} P(b_n(Y_n - y_n) \in S_\eta) \left[1 - \frac{4}{\pi\lambda_n\eta}\right]^k \\
\leq\ & \frac{\mu(S^\eta)}{\mu(S)} \frac{2}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(t)\, dt_1 \ldots dt_k \\
\leq\ & \frac{4}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(t)\, dt_1 \ldots dt_k. \qquad (2\text{-}17)
\end{aligned}$$

Since $\lambda_n \to \infty$ as $n \to \infty$ we can find an integer n_s so that

$$\sup_y \left[\frac{b_n^k}{\mu(S)} P(b_n(Y_n - y_n) \in S_\eta)\right] \leq M \qquad (2\text{-}18)$$

for $n \geq n_s$, where

$$M = \frac{5}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^*(t)\, dt_1 \ldots dt_k. \qquad (2\text{-}19)$$

This proves assertion (2-6). Note that $y \in S$ implies that $y - x \in S^\eta$ for $\|x\| \leq \eta$. Therefore for $n \geq n_s$ an upper bound for $m_n(y)$ is given by

$$\begin{aligned}
m_n(y) &= \frac{b_n^k}{\mu(S)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(b_n(Y_n - y + x) \in S)\, v_n(x)\, dx_1 \ldots dx_k \\
&\leq \frac{b_n^k}{\mu(S)} P(b_n(Y_n - y) \in S^\eta) \int \cdots \int_{\|x\| < \eta/b_n} v_n(x)\, dx_1 \ldots dx_k \\
&\quad + M \int \cdots \int_{\|x\| \geq \eta/b_n} v_n(x)\, dx_1 \ldots dx_k \\
&\leq \frac{b_n^k}{\mu(S)} P(b_n(Y_n - y_n) \in S^\eta) + M \left[1 - \left[1 - \frac{4}{\pi\lambda_n\eta}\right]^k\right]. \qquad (2\text{-}20)
\end{aligned}$$

Thus, from (2-14), (2-16) and (2-21) we get that

$$\limsup_n \left[\frac{b_n^k}{\mu(S)} P(b_n(Y_n - y_n) \in S_\eta)\right]$$
$$\leq f(y^*) \leq \left[\liminf_n \frac{b_n^k}{\mu(S)} P(b_n(Y_n - y_n) \in S^\eta)\right]. \tag{2-21}$$

Replacing S by S^η in the l.h.s. and S by S_η in the r.h.s. and using the relations $S \subset (S^\eta)_\eta$ and $(S_\eta)^\eta \subset S$ we get that

$$\limsup_n \left[\frac{b_n^k}{\mu(S^\eta)} P(b_n(Y_n - y_n) \in S)\right]$$
$$\leq f(y^*) \leq \liminf_n \left[\frac{b_n^k}{\mu(S_\eta)} P(b_n(Y_n - y_n) \in S)\right]. \tag{2-22}$$

Letting $\eta \to 0$ and using the fact $\mu(S^0) = \mu(\overline{S})$ we get the assertion (2-7). $\diamondsuit$

We have replaced the strong condition (2-3) of Theorem 2.1 with a weaker but more comlicated looking condition (2-4) in Theorem 2.2. The next theorem gives a convenient condition to verify condition (2-4).

THEOREM 2.3 *Let $\{Y_n, n \geq 1\}$ be a sequence of random vectors taking values in R^k. with c.f.'s $\{\hat{f}_n(t)\}$. Let $\{d_n\}$ be a sequence of real numbers such that $d_n \to \infty$. Assume that there exists $\delta > 0$ such that $g_n(t) = d_n^{-2} \log |\hat{f}_n(d_n t)|$ is differentiable in an open neighborhood $U = \{t : \|t\| < \delta\}$ of the origin for all $n \geq 1$. Suppose that there exists $\alpha > 0$ such that for $t \in U$,*

$$t'\nabla^2 g_n(t)t \quad \leq \quad -\alpha t't \tag{2-23}$$

for all $n \geq 1$. Then condition (2-4) of Theorem 2.2 is satisfied with $\beta_n = \delta d_n$.

Proof: An application of Taylor's theorem yields for $t \in U$.

$$\begin{aligned} g_n(t) &= g_n(0) + < t, \nabla g_n(0) > + \frac{t'\nabla^2 g_n(r_n)t}{2} \\ &= \frac{t'\nabla^2 g_n(r_n)t}{2} \leq -\frac{\alpha t't}{2}. \end{aligned} \quad (2\text{-}24)$$

where $r_n \in U$. Therefore for $t/d_n \in U$, we have,

$$g_n(t/d_n) \leq -\frac{\alpha t't}{2d_n^2}. \quad (2\text{-}25)$$

Let $\beta_n = \delta d_n$. Thus

$$\begin{aligned} |\hat{f}_n(t)| I(\|t\| < \beta_n) &\leq \exp(d_n^2(g_n(t/d_n))) \\ &\leq \exp(-\alpha t't/2) \end{aligned} \quad (2\text{-}26)$$

which is an integrable function. This completes the proof of the theorem. ◇

The next theorem obtains the limit of a function related to the Laplace transform of the positive part of Y_n when (2-6) and (2-7) hold. It plays an important role in the proofs of the strong large deviation theorem of Section 3.

THEOREM **2.4** *Let $\{Y_n,\ n \geq 1\}$ be a sequence of random vectors converging weakly to Y. Let $\{b_n\}$ be a sequence of real numbers such that $b_n \to \infty$. Let S be a measurable subset of R^k such that $0 < \mu(S^0) = \mu(\overline{S}) < \infty$. Let $q_n(y;\ b_n,\ S)$ be as defined in (2-1). Assume that $q_n(y;\ b_n,\ S)$ satisfies (2-6) and (2-7). Then*

$$b_n^k\ E[\exp(-b_n \sum_{i=1}^{k} Y_{ni})\ I(Y_n \geq 0)] \ \to\ f(0) \quad (2\text{-}27)$$

as $n \to \infty$.

Proof: Let $h > 0$ be given. Consider

$$I_n = E[\exp(-b_n \sum_{i=1}^{k} Y_{ni})\ I(Y_n \geq 0)]$$

$$= \sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\cdots\sum_{l_k=1}^{\infty} E\left[\exp(-b_n\sum_{i=1}^{k} Y_{ni}) \prod_{i=1}^{k} I\left(\frac{(l_i-1)h}{b_n} \le Y_{ni} < \frac{l_i h}{b_n}\right)\right] \quad (2\text{-}28)$$

Let $y_{nl_i} = (2l_i - 1)h/2b_n$. Then

$$I_n = \sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\cdots\sum_{l_k=1}^{\infty} E\left[\exp(-b_n\sum_{i=1}^{k} Y_{ni}) \prod_{i=1}^{k} I\left(\frac{-h}{2b_n} \le Y_{ni} - y_{nl_i} < \frac{h}{2b_n}\right)\right] \quad (2\text{-}29)$$

Let $m = m_h = [1/h^2]$ and $S_h = [-h/2,\, h/2]^k$, $l = (l_1, \ldots, l_k)$ and $y_{nl} = (y_{nl_1}, \ldots, y_{nl_k})$. We now get lower and upper bounds for I_n as follows:

$$\begin{aligned} I_n &\ge \sum_{l_1=1}^{m}\sum_{l_2=1}^{m}\cdots\sum_{l_k=1}^{m} \exp(-h\sum_{i=1}^{k} l_i)\, P\left(-\frac{h}{2b_n} \le Y_{ni} - y_{nl_i} < \frac{h}{2b_n},\ \forall\, i\right) \\ &= \frac{h^k}{b_n^k}\sum_{l_1=1}^{m}\sum_{l_2=1}^{m}\cdots\sum_{l_k=1}^{m} \exp\left(-h\sum_{i=1}^{k} l_i\right)\, q_n(y_{nl};\, b_n,\, S_h) \end{aligned} \quad (2\text{-}30)$$

and

$$\begin{aligned} I_n &\le \frac{h^k}{b_n^k}\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\cdots\sum_{l_k=1}^{\infty} \exp\left(-h\sum_{i=1}^{k}(l_i-1)\right)\, q_n(y_{nl};\, b_n,\, S_h) \\ &= \frac{h^k}{b_n^k}\sum_{l_1=1}^{m}\sum_{l_2=1}^{m}\cdots\sum_{l_k=1}^{m} \exp\left(-h\sum_{i=1}^{k}(l_i-1)\right)\, q_n(y_{nl};\, b_n,\, S_h) \\ &\quad + \frac{h^k}{b_n^k}\sum\sum\cdots\sum_{\{l\,:\,\|l\|>m\}} \exp\left(-h\sum_{i=1}^{k}(l_i-1)\right)\, q_n(y_{nl};\, b_n,\, S_h). \end{aligned} \quad (2\text{-}31)$$

Since $y_{nl} \to 0$ for each l we have $q_n(y_{nl};\, b_n,\, S_h) \to f(0)$ as $n \to \infty$. Thus, we get

$$\begin{aligned} \liminf_n\,(b_n^k\, I_n) &\ge f(0)\, h^k \left[\sum_{l=1}^{m} \exp(-lh)\right]^k \\ &= f(0)\left[\frac{h(\exp(-h) - \exp(-h(m+1)))}{1-\exp(-h)}\right]^k \end{aligned} \quad (2\text{-}32)$$

and

$$\limsup_n\,(b_n^k I_n) \le f(0)\,[h\textstyle\sum_{l=1}^{m}\exp(-(l-1)h)]^k + \frac{Mkh^k}{(1-\exp(-h))^{k-1}}\sum_{l=m+1}^{\infty}\exp(-(l-1)h)$$

$$= f(0)\left[\frac{h(1-\exp(-mh))}{1-\exp(-h)}\right]^k + Mk\left[\frac{h}{1-\exp(-h)}\right]^k \exp(-mh). \qquad (2\text{-}33)$$

From (2-33) and (2-34), letting $h \to 0$ and noting that $m = [1/h^2]$, we get

$$\lim_n (b_n^k I_n) \;=\; f(0). \qquad (2\text{-}34)$$

This completes the proof of the theorem. ♢

We will need the following extension of Theorem 2.4 in the next section.

THEOREM **2.5** *Let $\{Y_n, n \geq 1\}$ be a sequence of random vectors in R^k. Let $\hat{f}_n(t)$ be the c.f. of Y_n for $n \geq 1$. Let $\{\beta_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\beta_n \to \infty$ and $b_n \to \infty$. Assume that $\{Y_n, n \geq 1\}$ satisfies conditions (2-4) and (2-5) of Theorem 2.2. Let $\{P_n, n \geq 1\}$ be a sequence of real positive definite matrices whose eigenvalues are bounded above and bounded away from 0. Let $P_n^{1/2}$ be the unique positive definite square-root of P_n. Suppose that $P_n^{1/2} Y_n$ converges in distribution to a random vector Y. Then*

$$|P_n|^{1/2}\, b_n^k\, E[\exp(-b_n \sum_{i=1}^{k} Y_{ni})\; I(Y_n \geq 0)] \;\to\; f(0) \qquad (2\text{-}35)$$

as $n \to \infty$, where f is the p.d.f. of Y.

Proof: We will show that every subsequence has a further subsequence such that (2-36) holds for the subsequence. Consider a subsequence and denote it, for simplicity, as the sequence $\{n\}$ itself. Since the elements of P_n are uniformly bounded, we can find a further subsequence, which again for convenience is labelled as $\{n\}$, and a positive definite matrix P such that $P_n \to P$ as $n \to \infty$. Let $Q = P^{-1/2}$, then Y_n converges to QY in distribution. Note that the p.d.f. of QY at zero equals $|Q|\,f(0)$, where f is the p.d.f. of Y. The conclusion (2-36) now follows from Theorems 2.2 and 2.4. ♢

3 Strong Large Deviation Theorems.

Let $\{T_n, n \geq 1\}$ be a sequence of random vectors in R^k. Assume that the m.g.f. $\varphi_n(z) = E[\exp(< z, T_n >)]$ is holomorphic and nonvanishing in $\Omega^k(c)$, where $\Omega(c) = \{z \in \mathcal{C} : |z| < c\}$ for some $c > 0$ and $\mathcal{C}$ is the set of complex numbers. Let $\{a_n\}$ be a sequence of real numbers. Then

$$\psi_n(z) = \frac{1}{a_n} \log \phi_n(z) \tag{3-1}$$

is a well defined holomorphic function on $\Omega^k(c)$. Let $\nabla\psi_n(z) = (D_1\psi_n(z), \ldots, D_k\psi_n(z))$ be the vector of first order partial derivatives and $\nabla^2\psi_n(z)$ denote the matrix of second order partial derivatives, that is,

$$\nabla^2\psi_n(z) = (D_{ij}\psi_n(z)). \tag{3-2}$$

The determinant of the matrix $\nabla^2\psi_n(z)$ is denoted by $|\nabla^2\psi_n(z)|$. For u in R^k, let

$$\gamma_n(u) = \sup_{s \in I^k}[< s, u > -\psi_n(s)]. \tag{3-3}$$

Let $\{m_n\}$ be a bounded sequence of vectors in R^k such that there exists a sequence of vectors $\{\tau_n\}$ satisfying

$$\nabla\psi_n(\tau_n) = m_n \quad \text{and} \quad 0 < d < \tau_{ni} < b < c \quad \text{for all } 1 \leq i \leq k, \ n \geq 1. \tag{3-4}$$

Under these conditions we can see that $\gamma_n(m_n) =< m_n, \tau_n > -\psi_n(\tau_n)$. Let K_n be the d.f. of T_n. We will use the left continuous version of the distribution function which will enable us to write the identities in (3-6). Let

$$dH_n(y) = \frac{\exp(< y, \tau_n >)}{\phi_n(\tau_n)} dK_n(y), \quad y \in R^k \tag{3-5}$$

and T_n^* be random variable with d.f. $H_n(y)$. Let $T_n' = (T_n^* - a_n m_n)/\sqrt{a_n}$ and $Y_n = (Y_{n1}, \ldots, Y_{nk})$ where $Y_{ni} = \tau_{ni} T_{ni}'$. Using these new random variables we can write

$$\begin{aligned}
P(T_n \geq a_n m_n) &= \int_{\{y \in R^k :\, y \geq a_n m_n\}} dK_n(y) \\
&= \int_{\{y \in R^k : y \geq a_n m_n\}} \exp[- < y, \tau_n > + a_n \psi_n(\tau_n)]\, dH_n(y) \\
&= \exp(a_n \psi_n(\tau_n))\, E[\exp(-\sqrt{a_n} < \tau_n,\, T_n^* >)\, I(T_n^* \geq a_n m_n)] \\
&= \exp(-a_n \gamma_n(m_n))\, E[\exp(-\sum_{i=1}^{k} \tau_{ni}\, T_{ni}')\, I(T_n' \geq 0)] \\
&= \exp(-a_n \gamma_n(m_n))\, E[\exp(-b_n \sum_{i=1}^{k} Y_{ni})\, I(Y_n \geq 0)] \qquad (3\text{-}6)
\end{aligned}$$

where $b_n = \sqrt{a_n}$ and Y_n is as defined before. The next lemma obtains some important properties of the sequence $\{Y_n,\ n \geq 1\}$.

LEMMA **3.1** *Let $\{T_n, n \geq 1\}$ be a sequence of random vectors in R^k. Let $\{m_n\}$ be a bounded sequence of vectors in R^k such that there exist a sequence of vectors $\{\tau_n\}$ satisfying (3-4). Let the random vectors $\{Y_n, n \geq 1\}$ be as defined above. Assume that $\{T_n, n \geq 1\}$ satisfies the following conditions:*

(A) *There exists $0 < \beta < \infty$ such that $|\psi_n(z)| < \beta$ for all $n \geq 1$, $z \in \Omega^k(c)$.*

(B) *There exists $\alpha > 0$ such that the eigenvalues of $\nabla^2 \psi_n(\tau_n)$ are bounded below by α for all $n \geq 1$.*

Let D_n be the diagonal matrix with τ_{ni} as the ith diagonal element. Then, Y_n is asymptotically multivariate normal with mean 0 and variance-covariance matrix $D_n\, \nabla^2 \psi_n(\tau_n)\, D_n$.

Let $\hat{f}_n(t)$ be the c.f. of Y_n'. Then there exists $\delta > 0$ and $\alpha_1 > 0$, independent of n, such that

$$\sup_n |\hat{f}_n(t)| \, I(\|t\| < \delta\sqrt{a_n}) \leq \exp(-\alpha_1 t' t). \tag{3-7}$$

Proof: The c.f. of T_n' is given by

$$\hat{g}_n(t) = \exp(-i\sqrt{a_n} < t, m_n >) \, \frac{\phi_n(\tau_n + it/\sqrt{a_n})}{\phi_n(\tau_n)}. \tag{3-8}$$

Since $\psi_n(z)$ is a holomorphic function in $\Omega^k(c)$, for $|t| < (c-b)/2$ we have

$$\begin{aligned} \log \hat{g}_n(t) &= -i\sqrt{a_n} < t, m_n > + a_n[\psi_n(\tau_n + i\frac{t}{\sqrt{a_n}}) - \psi_n(\tau_n)] \\ &= -\frac{t'\nabla^2\psi_n(\tau_n)t}{2} + a_n \, R_n(\tau_n + i\frac{t}{\sqrt{a_n}}) \end{aligned} \tag{3-9}$$

where $R_n(\tau_n + it) = \sum_{j=3}^{\infty} \sum_{\{\alpha \,:\, |\alpha|=j\}} a_\alpha^{(n)}(it)^\alpha$. By Cauchy's theorem and condition **(A)** (see (3.7) of Chaganty and Sethuraman (1986)) we get the following bounds

$$|D_{ij}\psi_n(\tau_n)| \leq \frac{2\beta}{(c-b)^2}, \quad |a_\alpha^{(n)}| \leq \frac{\beta}{(c-b)^{|\alpha|}}. \tag{3-10}$$

Thus the elements of the matrix $\nabla^2\psi_n(\tau_n)$ are bounded uniformly in n. Now for $|t| < (c-b)/2$ and for all $n \geq 1$, we have

$$\begin{aligned} |R_n(\tau_n + it)| &\leq \sum_{j=3}^{\infty} \sum_{\{\alpha \,:\, |\alpha|=j\}} \prod_{i=1}^{k} |t_i|^{\alpha_i} \frac{\beta}{(c-b)^j} \\ &\leq \beta \sum_{j=3}^{\infty} \frac{|t|^j}{(c-b)^j} \, j^k. \end{aligned} \tag{3-11}$$

Hence

$$a_n |R_n(\tau_n + i\frac{t}{\sqrt{a_n}})| \leq \beta \, a_n \sum_{j=3}^{\infty} \frac{|t|^j}{(c-b)^j} \, \frac{j^k}{a_n^{j/2}}$$

$$\leq \beta a_n \sum_{j=3}^{\infty} \left(\frac{1}{2\sqrt{a_n}}\right)^j j^k \to 0 \tag{3-12}$$

as $n \to \infty$. From (3-9) and (3-12) we have that T_n' is asymptotically multivariate normal with mean 0 and covariance matrix $\nabla^2\psi_n(\tau_n)$. Let D_n be the diagonal matrix with τ_{ni} as the ith diagonal element. Note that $Y_n = D_n T_n'$ and Y_n is asymptotically multivariate normal with mean 0 and covariance matrix $D_n \nabla^2\psi_n(\tau_n) D_n$. We now proceed to show (3-7). Let

$$\begin{aligned} g_n(t) &= a_n^{-1} \log |\hat{g}_n(\sqrt{a_n}t)| \\ &= \text{Real } (\psi_n(\tau_n + it) - \psi_n(\tau_n)). \end{aligned} \tag{3-13}$$

We find that,

$$\begin{aligned} t' \nabla^2 g_n(t)\, t &= -t' \text{Real}\,(\nabla^2\psi_n(\tau_n + it))\, t \\ &= -t' \nabla^2\psi_n(\tau_n)\, t + \text{Real}(R_n(\tau_n + it)). \end{aligned} \tag{3-14}$$

From (3-11) we have $|R_n(\tau_n + it)|/|t|^2$ converges to 0 as $|t| \to 0$ uniformly in n. Therefore we can find a $\delta_1 > 0$ such that for $\|t\| < \delta_1$,

$$|R_n(\tau_n + it)| \leq \frac{\alpha}{2k} |t|^2 \leq \frac{\alpha}{2} t' t. \tag{3-15}$$

From (3-14), (3-15) and condition **(B)** we have

$$t' \nabla^2 g_n(t)\, t \leq -\alpha\, t'\, t + \frac{\alpha}{2} t'\, t = -\frac{\alpha}{2} t'\, t \tag{3-16}$$

for $\|t\| < \delta_1$. This shows that the c.f. $\hat{g}_n(t)$ satisfies condition (2-24) of Theorem 2.3. Hence

$$\sup_n |\hat{g}_n(t)|\, I(\|t\| < \delta_1 \sqrt{a_n}) \leq \exp(-\alpha t' t/2). \tag{3-17}$$

The c.f. $\hat{f}_n(t)$ of Y_n is given by $\hat{f}_n(t) = \hat{g}_n(D_n t)$. From (3-4) we get that $d\|t\| \le \|D_n t\| < a_0\|t\|$ and $d^2 t' t \le t' D_n D_n t \le a_0^2 t' t$. Let $\delta = \delta_1 / a_0$ and $\alpha_1 = \alpha d^2/2$. Then from (3-17) we get

$$\sup_n |\hat{f}_n(t)|\, I(\|t\| < \delta\sqrt{a_n}) \le \exp(-\alpha_1 t' t). \tag{3-18}$$

This completes the proof of the Lemma. ◇

LEMMA **3.2** *Let $\{T_n, n \ge 1\}$ be a sequence of random vectors in R^k. Let $\{m_n\}$ be a bounded sequence of vectors in R^k such that there exists a sequence of vectors $\{\tau_n\}$ satisfying (3-4). Let $a_n \to \infty$ be a sequence of real numbers and let $b_n = \sqrt{a_n}$. Suppose that T_n satisfies conditions* **(A)** *and* **(B)** *and the following condition:*

(C) *There exists $\delta_0 > 0$ such that*

$$\sup_{\{t:\delta<\|t\|\le\lambda\}} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| = o\left(\frac{1}{a_n^{k/2}}\right) \tag{3-19}$$

for any given δ and λ such that $0 < \delta < \delta_0 < \lambda$. Let Y_n be as defined in (3-6). Then

$$\prod_{i=1}^{k} \tau_{ni}\, |\nabla^2\psi_n(\tau_n)|^{1/2}\, b_n^k\; E[\exp(-b_n \sum_{i=1}^{k} Y_{ni})\, I(Y_n \ge 0)] \;\to\; \frac{1}{(2\pi)^{k/2}}. \tag{3-20}$$

Proof: We will prove this lemma by verifying the conditions of Theorem 2.5 for the sequence $\{Y_n\}$. From Lemma 3.1, Y_n is asymptotically multivariate normal with mean 0 and covariance matrix $D_n \nabla^2\psi_n(\tau_n) D_n$. From the same lemma, there exists a $\delta > 0$ such that (2-24) holds with $\beta_n = \delta b_n$. Using condition **(C)** we get that for fixed $\lambda > 0$,

$$\begin{aligned} \sup_{\delta b_n < \|t\| \le \lambda b_n} |\hat{f}_n(t)| &= \sup_{\delta b_n < \|t\| \le \lambda b_n} |\hat{g}_n(D_n t)| \\ &\le \sup_{\delta d b_n < \|t\| \le \lambda b b_n} |\hat{g}_n(t)| \end{aligned}$$

$$= \sup_{\delta d<\|t\|\leq\lambda b} \left|\frac{\phi_n(\tau_n+it)}{\phi_n(\tau_n)}\right| = o\left(\frac{1}{b_n^k}\right). \tag{3-21}$$

This verifies that the c.f. of Y_n also satisfies condition (2-25) of Theorem 2.2. From (3-4), (3-10) and condition **(B)** we can see that the eigenvalues of P_n are bounded above and away from zero below. Thus P_n satisfies the conditions in Theorem 2.5. The assertion (3-20) now follows from (2-36) of Theorem 2.5. ◇

REMARK **3.3** The conclusion (3-20) of Lemma 3.2 can also be rewritten in the form

$$b_n^k\ E[\exp(-b_n\sum_{i=1}^{k} Y_{ni})\, I(Y_n\geq 0)] \sim \frac{1}{(2\pi)^{k/2}\prod_{i=1}^{k}\ \tau_{ni}\ |\nabla^2\psi_n(\tau_n)|^{1/2}} \tag{3-22}$$

where the symbol $\sim$ means that the ratio of the two terms above tends to 1 as $n\rightarrow\infty$.

We are now in a position to state the main theorem of this section.

THEOREM **3.4** *Let $\{T_n, n\geq 1\}$ be a sequence of random vectors in R^k. Let $\{m_n\}$ be a bounded sequence of vectors in R^k such that there exists a sequence of vectors $\{\tau_n\}$ satisfying (3-4). Let $a_n\rightarrow\infty$ be a sequence of real numbers. Suppose that $\{T_n, n\geq 1\}$ satisfies conditions* **(A)**, **(B)** *and* **(C)**. *Then*

$$P\left(\frac{T_n}{a_n}\geq m_n\right) \sim \frac{\exp(-a_n\gamma_n(m_n))}{(2\pi a_n)^{k/2}\ \prod_{i=1}^{k}\tau_{ni}\ |\nabla^2\psi_n(\tau_n)|^{1/2}}. \tag{3-23}$$

Proof: Let Y_n be a function of T_n as defined in (3-6). The identity (3-6) states that

$$P(T_n\geq a_n m_n) = \exp(-a_n\gamma_n(m_n))\ E[\exp(-b_n\sum_{i=1}^{k} Y_{ni})\ I(Y_n\geq 0)] \tag{3-24}$$

Conditions **(A)**, **(B)** and **(C)** imply that $\{Y_n\}$ satisfies the conditions of Lemma 3.2. The conclusion (3-23) follows from (3-24) and (3-22). ◇

4 Applications.

In this section we will demonstrate the applicability of the theorems in Section 3 with an example. This example is concerned with approximating the probability of the tail areas of the multivariate F-distribution.

Let $G_{1n}, \ldots, G_{kn}$ and W_n be independent and distributed as chi-square with $n\lambda_1, \ldots, n\lambda_k$ and n degrees of freedom respectively, where $\lambda_1, \ldots, \lambda_k$ are fixed positive integers. Let $G_n = (G_{1n}, \ldots, G_{kn})$. The distribution of the random vector $X_n = G_n/W_n$ is known as the multivariate F-distribution with $(n\lambda_1, \ldots, n\lambda_k, n)$ degrees of freedom. Fix a vector $r_0 = (r_{01}, \ldots, r_{0k})$. We will use Theorem 3.4 to estimate the tail probability $P(X_n \geq r_0)$, which can be rewritten as

$$\begin{aligned} P(X_n \geq r_0) &= P(G_n - r_0 W_n \geq 0) \\ &= P(T_n \geq 0) \end{aligned} \tag{4-1}$$

where $T_n = G_n - r_0 W_n$. The m.g.f. of T_n is given by

$$\phi_n(z) = \prod_{i=1}^{k} \left(\frac{1}{1-2z_i}\right)^{\lambda_i n/2} \left(\frac{1}{1+2\sum_{i=1}^{k} z_i r_{0i}}\right)^{n/2}, \quad z \in \Omega^k \tag{4-2}$$

where $\Omega = \{z \in \mathcal{C} : \text{Real}(z) < 1/2\}$. We will verify that $\{T_n\}$ satisfies the conditions **(A)**, **(B)** and **(C)** with $a_n = n$ and $m_n = 0$ for $n \geq 1$. Note that $\psi_n(z) = \frac{1}{n} \log \phi_n(z)$ is independent of n and is given by

$$\psi(z) = -\frac{1}{2}\sum_{i=1}^{k} \lambda_i \log(1-2z_i) - \frac{1}{2}\log(1+2\sum_{i=1}^{k} z_i r_{0i}), \tag{4-3}$$

and $\psi(z)$ is bounded for $\Omega^k(c)$ for any $0 < c < 1/2$. Thus $\{T_n\}$ satisfies condition **(A)**. Since $m_n = 0$ and $\psi_n = \psi$ are independent of n, we have τ_n's satisfying (3-4) do not depend on n and are equal to τ_0, where the ith component of τ_0 is given by

$$\tau_{0i} = \frac{1}{2}\left[1 - \frac{\lambda_i(1+k\overline{r})}{r_{0i}(1+k\overline{\lambda})}\right], \quad 1 \le i \le k. \tag{4-4}$$

Assume that the vector $r_0 = (r_{01}, \ldots, r_{0k})$ satisfies

$$\frac{r_{0i}}{1+k\,\overline{r}} > \frac{\lambda_i}{1+k\,\overline{\lambda}} \quad \text{for all} \quad 1 \le i \le k \tag{4-5}$$

where $\overline{r} = \sum r_{0i}/k$ and $\overline{\lambda} = \sum \lambda_i/k$. From (4-4) and (4-5) we can verify that there exists $d > 0$ and $0 < b < 1/2$ such that $d < \tau_{0i} < b$ for all $1 \le i \le k$. Thus $\tau_n = \tau_0$ satisfies (3-4). With some simple calculations we can show that the determinant of $\nabla^2\psi(\tau_0)$ is given by

$$|\nabla^2\psi(\tau_0)| = \prod_{i=1}^{k}\left(\frac{2r_{0i}^2}{\lambda_i}\right)\left(\frac{1+k\,\overline{\lambda}}{1+k\,\overline{r}}\right)^{2k}(1+k\,\overline{\lambda}) \tag{4-6}$$

showing that the eigenvalues of $\nabla^2\psi(\tau_0)$ are bounded below away from zero. Thus $\{T_n\}$ satisfies condition **(B)**. Using (4-2) we can see that for any given $\delta > 0$ and $\lambda > \delta$,

$$\sup_{\{t\,:\,\delta<\|t\|\le\lambda\}}\left|\frac{\phi_n(\tau_0+it)}{\phi_n(\tau_0)}\right| \le \sup_{\{t\,:\,\delta<\|t\|\le\lambda\}}\left|\frac{\phi_0(\tau_0+it)}{\phi_0(\tau_0)}\right|^n \tag{4-7}$$

where $\phi_0(z) = \prod_{i=1}^{k}(1-2z_i)^{-\lambda_i/2}$ is the joint m.g.f. of k independent chi-square random variables. Since $\phi_0(\tau_0+it)/\phi_0(\tau_0)$ is the c.f. of a nonlattice random vector, there exists a $\eta > 0$ such that

$$\sup_{\{t\,:\,\delta<\|t\|\le\lambda\}} \left| \frac{\phi_0(\tau_0+it)}{\phi_0(\tau_0)} \right| < e^{-\eta}. \tag{4-8}$$

From (4-7) and (4-8) we get that

$$\sup_{\{t\,:\,\delta<\|t\|\le\lambda\}} \left| \frac{\phi_0(\tau_0+it)}{\phi_0(\tau_0)} \right| = o\left(\frac{1}{n^{k/2}}\right). \tag{4-9}$$

Thus we have verified that $\{T_n\}$ satisfies condition **(C)**. We now can apply Theorem 3.4 to the sequence $\{T_n,\, n \ge 1\}$ with the choice of $a_n = n$ and $m_n = 0$. Notice that $\gamma_n(0) = -\psi(\tau_0)$, where ψ is defined by (4-3). Thus from the conclusion (3-23) of Theorem 3.4 and (4-1) we get

$$P(X_n \ge r_0) \sim \frac{\exp(n\,\psi(\tau_0))}{(2\pi\, n)^{k/2} \prod_{i=1}^{k} \tau_{0i}\, |\nabla^2\psi(\tau_0)|^{1/2}} \tag{4-10}$$

for any r_0 satisfying (4-5), where ψ, τ_0 and $|\nabla^2\psi(\tau_0)|$ are given by (4-3), (4-4) and (4-6) respectively.

Another application of Theorem 3.4 can be given for tail probabilities of a multinomial distribution. We do not present the details of this application since it is very similar to the above application to the multivariate F-distribution.

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September 16, 1992

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Dear Professor Rao:

I am submitting three copies of the manuscript "Multidimensional Strong Large Deviation Theorems" with Narasinga R. Chaganty for consideration for publication in the Journal of Multivariate Analysis. A copy of the manuscript and this letter are also being sent to the chief editor.

Sincerely yours

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